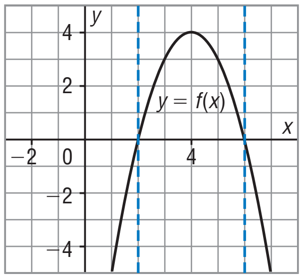
Lesson 8.5 Exercises, pages 680–688

A

3. For each graph of $y = f(x)$, draw vertical lines to represent the vertical asymptotes, if they exist, of the graph of $y = \frac{1}{f(x)}$.

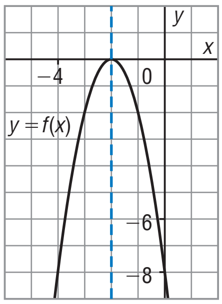
a)



The x-intercepts of the graph of $y = f(x)$ are 2 and 6.

So, the graph of $y = \frac{1}{f(x)}$ has vertical asymptotes at $x = 2$ and $x = 6$.

b)



The x-intercept of the graph of $y = f(x)$ is -2. So, the graph of $y = \frac{1}{f(x)}$ has a vertical asymptote at $x = -2$.

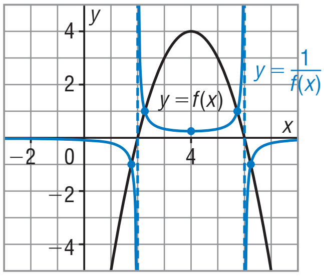
4. For each graph in question 3, identify the values of x for which the graph of $y = \frac{1}{f(x)}$ is above the x-axis, and for which it is below the x-axis.

a) Above the x-axis: $2 < x < 6$
Below the x-axis:
$x < 2$ or $x > 6$

b) Below the x-axis:
$x \in \mathbb{R}, x \neq -2$
Above the x-axis: never

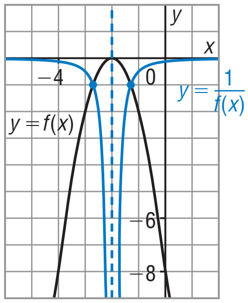
5. On each graph from question 3, sketch the graph of $y = \dfrac{1}{f(x)}$.

a)



The x-axis is a horizontal asymptote. Plot points where the lines $y = 1$ and $y = -1$ intersect the graph. These points are common to both graphs. The graph of $y = f(x)$ has vertex (4, 4), so point $\left(4, \frac{1}{4}\right)$ lies on $y = \frac{1}{f(x)}$. Since the graph has 2 vertical asymptotes, it has Shape 3.

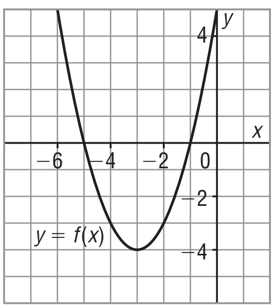
b)



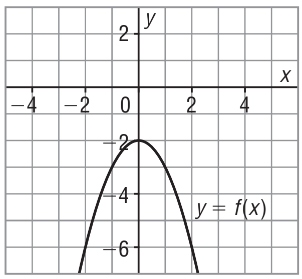
The x-axis is a horizontal asymptote. Plot points where the line $y = -1$ intersects the graph. These points are common to both graphs. Since the graph has one vertical asymptote, it has Shape 2.

6. Match each graph of $y = f(x)$ to the corresponding graph of $y = \dfrac{1}{f(x)}$. What features of the graphs did you use to make your decisions?

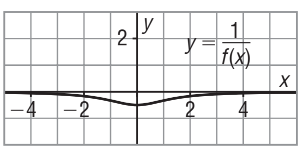
a)



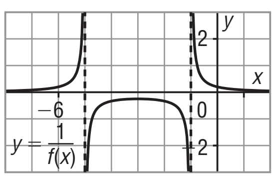
b)



i)



ii)



The graph that corresponds to graph a is graph ii. The graph of $y = f(x)$ in part a has 2 x-intercepts, so the graph of its reciprocal function has 2 vertical asymptotes.
The graph that corresponds to graph b is graph i. The graph of $y = f(x)$ in part b has no x-intercepts, so the graph of its reciprocal function has no vertical asymptotes.

B

7. How many vertical asymptotes does the graph of each reciprocal function have? Identify the equation of each vertical asymptote.

a) $y = \dfrac{1}{-(x + 3)^2}$

$-(x + 3)^2 = 0$ when $x = -3$. So, the graph of $y = \dfrac{1}{-(x + 3)^2}$ has 1 vertical asymptote, $x = -3$.

b) $y = \dfrac{1}{(x - 2)(x + 6)}$

$(x - 2)(x + 6) = 0$ when $x = 2$ or $x = -6$. So, the graph of $y = \dfrac{1}{(x-2)(x + 6)}$ has 2 vertical asymptotes, $x = 2$ and $x = -6$.

c) $y = \dfrac{1}{x^2 + x - 6}$

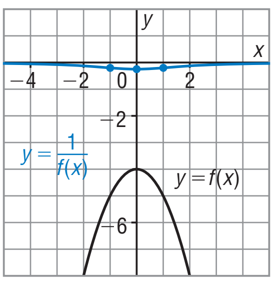
$x^2 + x - 6 = (x + 3)(x - 2)$
$(x + 3)(x - 2) = 0$ when $x = -3$ or $x = 2$. So, the graph of $y = \dfrac{1}{x^2 + x - 6}$ has 2 vertical asymptotes, $x = -3$ and $x = 2$.

d) $y = \dfrac{1}{-3x^2 - 9}$

$-3x^2 - 9 = -3(x^2 + 3)$
Since $x^2 + 3$ cannot be 0, the graph of $y = \dfrac{1}{-3x^2 - 9}$ has no vertical asymptotes.

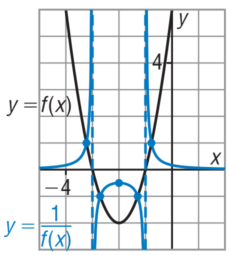
8. On the graph of each quadratic function $y = f(x)$, sketch a graph of the reciprocal function $y = \dfrac{1}{f(x)}$. Identify the vertical asymptotes, if they exist. Explain your strategies.

a)

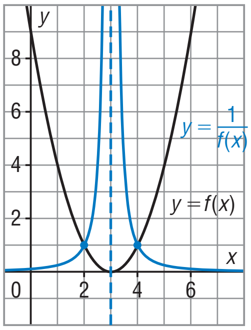


The graph of $y = f(x)$ has no x-intercepts, so the graph of $y = \dfrac{1}{f(x)}$ has no vertical asymptotes and has Shape 1.
Horizontal asymptote: $y = 0$
Points $(0, -4)$, $(-1, -5)$, and $(1, -5)$ lie on $y = f(x)$, so points $(0, -0.25)$, $(-1, -0.2)$, and $(1, -0.2)$ lie on $y = \dfrac{1}{f(x)}$.

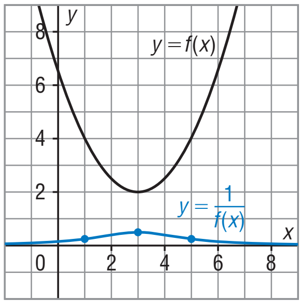
b)



The graph of $y = f(x)$ has 2 x-intercepts, so the graph of $y = \dfrac{1}{f(x)}$ has 2 vertical asymptotes, $x = -3$ and $x = -1$, and has Shape 3.
Horizontal asymptote: $y = 0$
Plot points where the lines $y = 1$ and $y = -1$ intersect the graph of $y = f(x)$. These points are common to both graphs. The graph of $y = f(x)$ has vertex $(-2, -2)$, so point $(-2, -0.5)$ lies on $y = \dfrac{1}{f(x)}$.

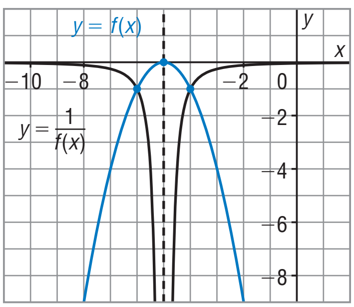
c) 

The graph of $y = f(x)$ has 1 x-intercept, so the graph of $y = \frac{1}{f(x)}$ has 1 vertical asymptote, $x = 3$, and has Shape 2.
Horizontal asymptote: $y = 0$
Plot points where the line $y = 1$ intersects the graph of $y = f(x)$. These points are common to both graphs.

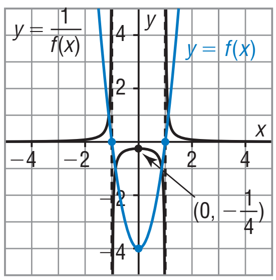
d) 

The graph of $y = f(x)$ has no x-intercepts, so the graph of $y = \frac{1}{f(x)}$ has no vertical asymptotes and has Shape 1.
Horizontal asymptote: $y = 0$
Points (3, 2), (1, 4), and (5, 4) lie on $y = f(x)$, so points (3, 0.5), (1, 0.25), and (5, 0.25) lie on $y = \frac{1}{f(x)}$.

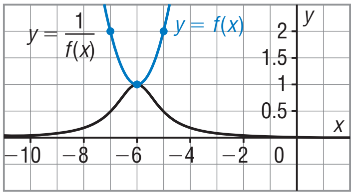
9. On the graph of each reciprocal function $y = \frac{1}{f(x)}$, sketch a graph of the quadratic function $y = f(x)$.

a) 

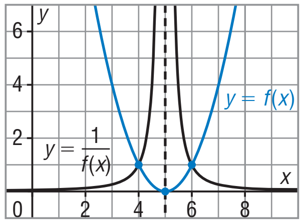
The graph has one vertical asymptote, $x = -5$, so the graph of $y = f(x)$ has vertex $(-5, 0)$. The line $y = -1$ intersects the graph at 2 points that are common to both graphs.

b) 

The graph has 2 vertical asymptotes, $x = -1$ and $x = 1$, so the graph of $y = f(x)$ has x-intercepts -1 and 1.
The point $\left(0, -\frac{1}{4}\right)$ is on the line of symmetry, so $(0, -4)$ is the vertex of $y = f(x)$.

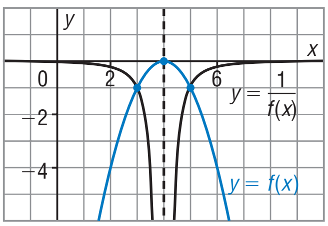
c) 

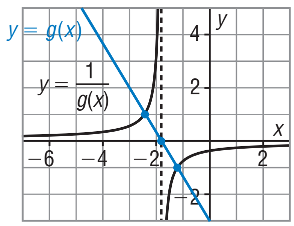
The graph has no vertical asymptote, so the graph of $y = f(x)$ has no x-intercept. The point $(-6, 1)$ is on the line of symmetry, so $(-6, 1)$ is the vertex of $y = f(x)$. Points $(-7, 0.5)$ and $(-5, 0.5)$ lie on $y = f(x)$, so points $(-7, 2)$ and $(-5, 2)$ lie on $y = \frac{1}{f(x)}$.

d) 

The graph has one vertical asymptote, $x = 5$, so the graph of $y = f(x)$ has vertex $(5, 0)$. The line $y = 1$ intersects the graph at 2 points that are common to both graphs.

10. The graphs of the reciprocal functions $y = \frac{1}{f(x)}$ and $y = \frac{1}{g(x)}$ are shown below.

i) 

ii) 

a) Identify whether each function $y = f(x)$ and $y = g(x)$ is linear or quadratic. How did you decide?

i) $y = f(x)$ is quadratic. I recognize the shape of the graph as that of the reciprocal of a quadratic function with 1 x-intercept.

ii) $y = g(x)$ is linear. I recognize the shape of the graph as that of the reciprocal of a linear function.

b) On the graph of each reciprocal function, sketch a graph of the related linear or quadratic function. What strategy did you use each time?

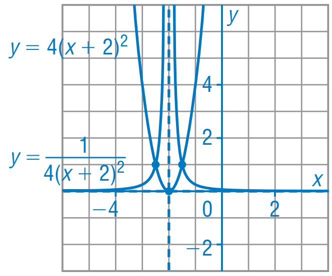
i) The graph has one vertical asymptote, $x = 4$, so the graph of $y = f(x)$ has vertex $(4, 0)$. The line $y = -1$ intersects the graph at 2 points that are common to both graphs.

ii) Vertical asymptote is about $x = -1.8$, so graph of $y = g(x)$ has x-intercept -1.8. Mark points at $y = 1$ and $y = -1$ on graph of $y = \frac{1}{g(x)}$, then draw a line through these points for the graph of $y = g(x)$.

11. Graph each pair of functions on the same grid. Explain your strategies.

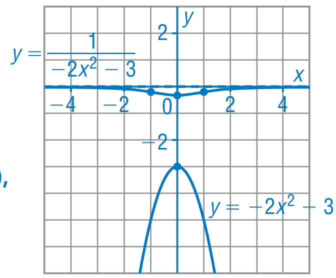
a) $y = 4(x + 2)^2$ and $y = \dfrac{1}{4(x + 2)^2}$

The graph of $y = 4(x + 2)^2$ opens up, has vertex $(-2, 0)$, and x-intercept -2.
The graph of $y = \dfrac{1}{4(x + 2)^2}$ has vertical asymptote, $x = -2$ and horizontal asymptote $y = 0$. Plot points where the line $y = 1$ intersects the graph of $y = 4(x + 2)^2$. These points are common to both graphs. The graph of the reciprocal function has Shape 2.



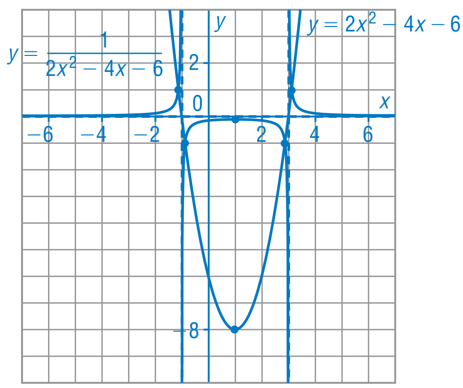
b) $y = -2x^2 - 3$ and $y = \dfrac{1}{-2x^2 - 3}$

The graph of $y = -2x^2 - 3$ opens down, with vertex $(0, -3)$, so the graph has no x-intercepts. The graph of $y = \dfrac{1}{-2x^2 - 3}$ has no vertical asymptotes; the horizontal asymptote is $y = 0$. Points $(0, -3)$, $(1, -5)$, and $(-1, -5)$ lie on $y = -2x^2 - 3$. So, points $\left(0, -\dfrac{1}{3}\right)$, $(1, -0.2)$, and $(-1, -0.2)$ lie on $y = \dfrac{1}{-2x^2 - 3}$. The graph of the reciprocal function has Shape 1.



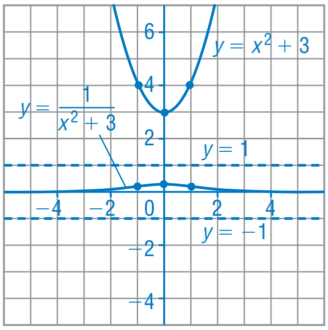
c) $y = 2x^2 - 4x - 6$ and $y = \dfrac{1}{2x^2 - 4x - 6}$

The graph of $y = 2x^2 - 4x - 6$, or $y = 2(x - 3)(x + 1)$ opens up, has x-intercepts 3 and -1, and vertex $(1, -8)$. The graph of $y = \dfrac{1}{2x^2 - 4x - 6}$ has vertical asymptotes $x = 3$ and $x = -1$, and a horizontal asymptote $y = 0$. Plot points where the lines $y = 1$ and $y = -1$ intersect the graph of $y = 2(x - 3)(x + 1)$. These points are common to both graphs. Point $(1, -8)$ lies on $y = 2x^2 - 4x - 6$, so point $\left(1, -\dfrac{1}{8}\right)$ lies on $y = \dfrac{1}{2x^2 - 4x - 6}$. The graph of the reciprocal function has Shape 3.



12. A quadratic function and its reciprocal are graphed on a grid. The horizontal lines $y = 1$ and $y = -1$ do not intersect either graph. Sketch a possible graph of the quadratic function and its reciprocal. Explain your strategy.

If the lines $y = 1$ and $y = -1$ do not intersect the graph of a quadratic function, the vertex of the graph either lies above the line $y = 1$ and opens up, or it lies below the line $y = -1$ and opens down. In either case, the graph of the quadratic function has no x-intercepts and the graph of the corresponding reciprocal function has no vertical asymptotes. Here is the graph of $y = x^2 + 3$ and $y = \frac{1}{x^2 + 3}$.



13. A reciprocal function has the form $y = \frac{1}{ax^2 + b}$, $a \neq 0$, $b \neq 0$. How can you determine the number of asymptotes of the graph of $y = \frac{1}{ax^2 + b}$ when:

a) both a and b are positive?

Look at the quadratic function $y = ax^2 + b$. When both a and b are positive, the graph opens up and its vertex is above the x-axis. So, the graph of $y = ax^2 + b$ has no x-intercepts and the graph of $y = \frac{1}{ax^2 + b}$ has no vertical asymptotes.

b) both a and b are negative?

Look at the quadratic function $y = ax^2 + b$. When both a and b are negative, the graph opens down and its vertex is below the x-axis. So, the graph of $y = ax^2 + b$ has no x-intercepts and the graph of $y = \frac{1}{ax^2 + b}$ has no vertical asymptotes.

c) a and b have opposite signs?

Look at the quadratic function $y = ax^2 + b$. When a and b have opposite signs, the graph either opens up with its vertex below the x-axis, or it opens down with its vertex above the x-axis. So, the graph of $y = ax^2 + b$ has 2 x-intercepts and the graph of $y = \frac{1}{ax^2 + b}$ has 2 vertical asymptotes.

C

14. The reciprocal function $y = \frac{1}{px^2 + (2p + 1)x + p}$ has one vertical asymptote. Show that $p = -\frac{1}{4}$.

If the reciprocal function has one vertical asymptote, then the related quadratic function has one x-intercept; that is, the quadratic equation $px^2 + (2p + 1)x + p = 0$ has equal roots.
The equation has equal roots when $b^2 - 4ac = 0$.
Substitute: $b = 2p + 1$, $a = p$, $c = p$

$$(2p + 1)^2 - 4(p)(p) = 0$$
$$4p^2 + 4p + 1 - 4p^2 = 0$$
$$4p + 1 = 0$$
$$p = -\frac{1}{4}$$

15. Determine the values of k for which the reciprocal function $y = \frac{1}{x^2 + kx + 4}$ has:

a) no vertical asymptotes

The reciprocal function has no vertical asymptotes, so the related quadratic function has no x-intercepts; that is, $x^2 + kx + 4 = 0$ has no real roots. The equation has no real roots when $b^2 - 4ac < 0$.
Substitute: $a = 1$, $b = k$, $c = 4$

$$k^2 - 4(1)(4) < 0$$
$$k^2 - 16 < 0$$
$$k^2 < 16$$
$$-4 < k < 4$$

b) one vertical asymptote

The reciprocal function has one vertical asymptote, so the related quadratic function has one x-intercept; that is, $x^2 + kx + 4 = 0$ has equal roots. The equation has equal roots when $b^2 - 4ac = 0$.
Substitute: $a = 1$, $b = k$, $c = 4$

$$k^2 - 4(1)(4) = 0$$
$$k^2 - 16 = 0$$
$$k^2 = 16$$
$$k = 4 \text{ or } k = -4$$

c) two vertical asymptotes

The reciprocal function has two vertical asymptotes, so the related quadratic function has two x-intercepts; that is, $x^2 + kx + 4 = 0$ has 2 real roots. The equation has 2 real roots when $b^2 - 4ac > 0$.
Substitute: $a = 1$, $b = k$, $c = 4$

$$k^2 - 4(1)(4) > 0$$
$$k^2 - 16 > 0$$
$$k^2 > 16$$
$$k < -4 \text{ or } k > 4$$

16. Determine the equation of each quadratic function $y = f(x)$ you graphed in question 9. Describe your strategies.

a) The graph has vertex $(-5, 0)$ and passes through the point $(-6, -1)$.
So, the equation has the form: $y = a(x + 5)^2$
Substitute: $x = -6$, $y = -1$
$-1 = a(-6 + 5)^2$
$-1 = a$
The equation of the quadratic function is $y = -(x + 5)^2$.

b) The graph has vertex $(0, -4)$ and passes through the point $(1, 0)$.
So, the equation has the form: $y = ax^2 - 4$
Substitute: $x = 1$, $y = 0$
$0 = a(1)^2 - 4$
$4 = a$
The equation of the quadratic function is $y = 4x^2 - 4$.

c) The graph has vertex $(-6, 1)$ and passes through the point $(-5, 2)$.
So, the equation has the form: $y = a(x + 6)^2 + 1$
Substitute: $x = -5$, $y = 2$
$2 = a(-5 + 6)^2 + 1$
$1 = a$
The equation of the quadratic function is $y = (x + 6)^2 + 1$.

d) The graph has vertex $(5, 0)$ and passes through the point $(4, 1)$.
So, the equation has the form: $y = a(x - 5)^2$
Substitute: $x = 4$, $y = 1$
$1 = a(4 - 5)^2$
$1 = a$
The equation of the quadratic function is $y = (x - 5)^2$.